Time Dependent Entropy of Constant Force Motion

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Abstract

Time dependent entropy of constant force motion is investigated. Their joint entropy so called Leipnik's entropy is obtained. The main purpose of this work is to calculate Leipnik's entropy by using time dependent wave function which is obtained by the Feynman path integral method. It is found that, in this case, the Leipnik's entropy increase with time and this result has same behavior free particle case.

Keywords: Path integral, joint entropy, constant force motion.

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I. INTRODUCTION

The information entropy plays a major role in a stronger formulation of the uncertainty relations [1]. This relation may be mathematically defined by using the Boltzmann-Shannon information entropy and the von Neumann entropy. In the literature for both open and closed quantum systems, the different information-theoretical entropy measures have been discussed [3, 4, 5]. In contrast, the joint entropy [6, 7] can also be used to properties the loss of information, related to evolving pure quantum states [8]. The joint entropy of the physical systems were conjectured by Dunkel and Trigger [9] in which their systems named MACS (maximal classical states). The Leibnik entropy of the simple harmonic oscillator was determined not monotonically increase with time [10]. In this work, we give a uniform description of the complete joint entropy information of system in motion under a constant force.

This paper is organized as follows. In section II, we explain fundamental definitions needed for the calculation. In section III, we deal with calculation and results for constant force systems. Moreover, we obtain the analytical solution of Kernel, wave function in both coordinate and momentum space and its joint entropy. We also obtain same quantities for constant magnetic field case. Finally, we present the conclusion in section IV.

II. FUNDAMENTAL DEFINITIONS

We consider a classical system with $d = sN$ degrees of freedom, where N is the particle number and s is number of spatial dimensions [9]. Apart from this, let us describe $g(x, p, t) = g(x_1, ..., x_n, p_1, ..., p_d, t)$ which is non-negative, time dependent phase space density function of system. The density function is assuming to be normalized to unity,

$$\int dxdpg(x, p, t) = 1. \tag{1}$$

The Gibbs-Shannon entropy is described by

$$S(t) = -\frac{1}{N!} \int dxdpg(x, p, t) ln(h^d g(x, p, t)), \tag{2}$$

where $h = 2\pi\hbar$ is the Planck constant. Schrödinger wave equation with the Born interpretation [11] is given by

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi. \tag{3}$$

The quantum probability densities are defined in position and momentum spaces as $|\psi(x,t)|^2$ and $|\tilde{\psi}(p,t)|^2$, where $|\tilde{\psi}(p,t)|^2$ is given as

$$\tilde{\psi}(p,t) = \int \frac{dx e^{-ipx/\hbar}}{(2\pi\hbar)^{d/2}}\psi(x,t). \tag{4}$$

Leipnik proposed the product function as [9]

$$g_j(x,p,t) = |\psi(x,t)|^2|\tilde{\psi}(p,t)|^2 \geq 0. \tag{5}$$

Substituting Eq. (5) into Eq. (2), we get the joint entropy $S_j(t)$ for the pure state $\psi(x,t)$ or equivalently can be written in the following form [9]

$$\begin{aligned} S_j(t) &= -\int dx|\psi(x,t)|^2\ln|\psi(x,t)|^2 - \int dp|\tilde{\psi}(p,t)|^2\ln|\tilde{\psi}(p,t)|^2 - \\ &- \ln h^d. \end{aligned} \tag{6}$$

We find time dependent wave function by means of the Feynman path integral which has form [12]

$$\begin{aligned} K(x'',t'';x',t') &= \int_{x'=x(t')}^{x''=x(t'')} Dx(t)e^{\frac{i}{\hbar}S[x(t)]} \\ &= \int_{x'}^{x''} Dx(t)e^{\frac{i}{\hbar}\int_{t'}^{t''}L[x,\dot{x},t]dt}. \end{aligned} \tag{7}$$

The Feynman kernel can be related to the time dependent Schrödinger's wave function

$$K(x'',t'';x',t') = \sum_{n=0}^{\infty}\psi_n^*(x',t')\psi_n(x'',t''). \tag{8}$$

The propagator in semiclassical approximation reads

$$K(x'',t'';x',t') = \Big[\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial x'\partial x''}S_{cl}(x'',t'';x',t')\Big]^{1/2}e^{\frac{i}{\hbar}S_{cl}(x'',t'';x',t')}. \tag{9}$$

The prefactor is often referred to as the Van Vleck-Pauli-Morette determinant [13, 14]. The $F(x'',t'';x',t')$ is given by

$$F(x'',t'';x',t') = \Big[\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial x'\partial x''}S_{cl}(x'',t'';x',t')\Big]^{1/2}. \tag{10}$$

III. CALCULATION AND RESULTS

A. Constant Force

The Lagrangian for present case is

$$L(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 + fx \tag{11}$$

The classical path obeys

$$m\ddot{x}_{cl} = f \tag{12}$$

The solution of above equation is

$$x_{cl}(\tau) = x_0 + \Big(\frac{x - x_0}{t - t_0} - \frac{1}{2}\frac{f}{m}(t - t_0)\Big)\tau + \frac{1}{2}\frac{f}{m}\tau^2 \tag{13}$$

One obtains for classical action integral along the classical path [12]

$$S(x_{cl}(\tau)) = \frac{1}{2}m\frac{(x - x_0)^2}{t - t_0} + \frac{1}{2}(x + x_0)f(t - t_0) - \frac{1}{24}\frac{f^2}{m}(t - t_0)^3 \tag{14}$$

and finally, for the kernel

$$\begin{aligned} K(x'', x'; T) \;=\; & \Big[\frac{m}{2\pi i\hbar T}\Big]^{1/2} \exp\Big[\frac{im}{2\hbar}\frac{(x - x_0)^2}{T} + \frac{i}{2\hbar}(x + x_0)fT - \\ & - \frac{i}{24\hbar}\frac{f^2}{m}T^3\Big] \end{aligned} \tag{15}$$

The dependent wave function at time $t > 0$

$$\begin{aligned} \Psi(x, t) \;=\; & \Big[\frac{1 - i\frac{\hbar t}{m\sigma^2}}{1 + i\frac{\hbar t}{m\sigma^2}}\Big]^{1/4}\Big[\frac{1}{\pi\sigma^2(1 + \frac{\hbar^2 t^2}{m^2\sigma^4})}\Big]^{1/4} \exp\Big[-\frac{(x - \frac{p_0}{m}t - \frac{ft^2}{2m})^2}{2\sigma^2(1 + \frac{\hbar^2 t^2}{m^2\sigma^4})} \times \\ & \times\; (1 - i\frac{\hbar t}{m\sigma^2})\Big] \exp\Big[\frac{i}{\hbar}(p_0 + ft)x - \frac{i}{\hbar}\int_0^t d\tau \frac{(p_0 + f\tau)^2}{2m}\Big] \end{aligned} \tag{16}$$

where σ is width of Gaussian curve. The corresponding probability distribution is

$$|\Psi(x, t)|^2 = \Big[\frac{1}{\pi\sigma^2(1 + \frac{\hbar^2 t^2}{m^2\sigma^4})}\Big]^{1/2} \exp\Big[-\frac{(x - \frac{p_0 t}{m} - \frac{ft^2}{2m})^2}{\sigma^2(1 + \frac{\hbar^2 t^2}{m^2\sigma^4})}\Big] \tag{17}$$

or

$$|\Psi(x, t)|^2 = \Big[\frac{1}{\pi\sigma^2(1 + \frac{\hbar^2 t^2}{m^2\sigma^4})}\Big]^{1/2} \exp\Big[-\frac{(x - \frac{p_0 t}{m} - \frac{ft^2}{2m})^2}{\sigma^2(1 + \frac{\hbar^2 t^2}{m^2\sigma^4})}\Big] \tag{18}$$

The probability density in coordinate space is shown Fig.1. The probability density in momentum space can be written easily

$$|\Psi(p,t)|^2 = \left[\frac{\sigma^2}{\pi\hbar^2}\right]^{1/2} \exp\left[\frac{-\sigma^2}{\hbar^2}(p+(p_0+ft))^2\right] \tag{19}$$

The time dependent joint entropy can be obtained from Eq. 6 as

$$S_j(t) = \ln(\frac{e}{2})\sqrt{1+\frac{\hbar^2 t^2}{m^2\sigma^4}} \tag{20}$$

The joint entropy of this system is shown Fig.2 and Fig.3. It is important that Eq. 20 is in agreement with following general inequality for the joint entropy:

$$S_j(t) \geq \ln(\frac{e}{2}) \tag{21}$$

originally derived by Leipnik for arbitrary one-dimensional one-particle wave functions.

IV. CONCLUSION

We have investigated the joint entropy for constant for motion. We have obtained the time dependent wave function by means of Feynman Path integral technique. In this case, we have found that the joint entropy increase with time and the results harmony prior studied. The joint entropy has same behavior as free particle case. This result indicates that the information entropy is getting increase with time.

V. ACKNOWLEDGEMENTS

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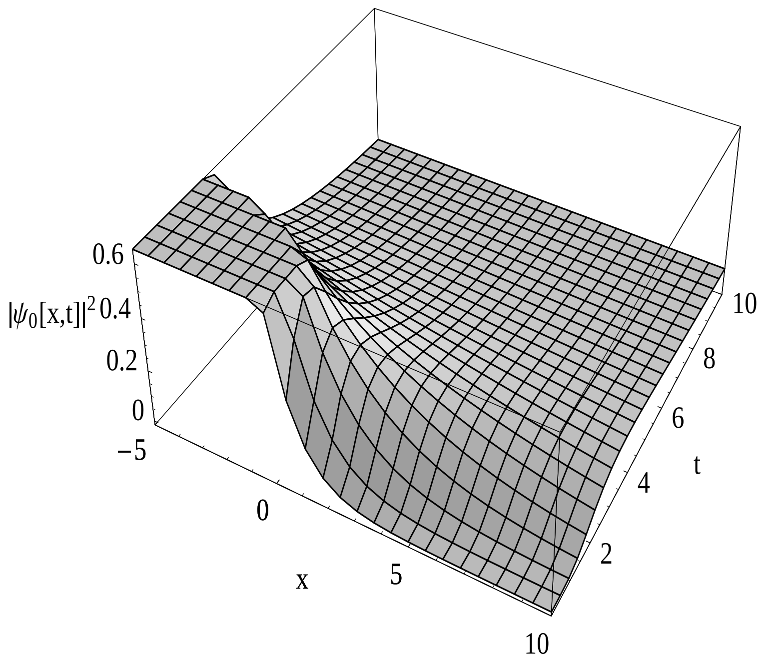

FIG. 1: $|\Psi(x,t)|^2$ versus time and coordinate

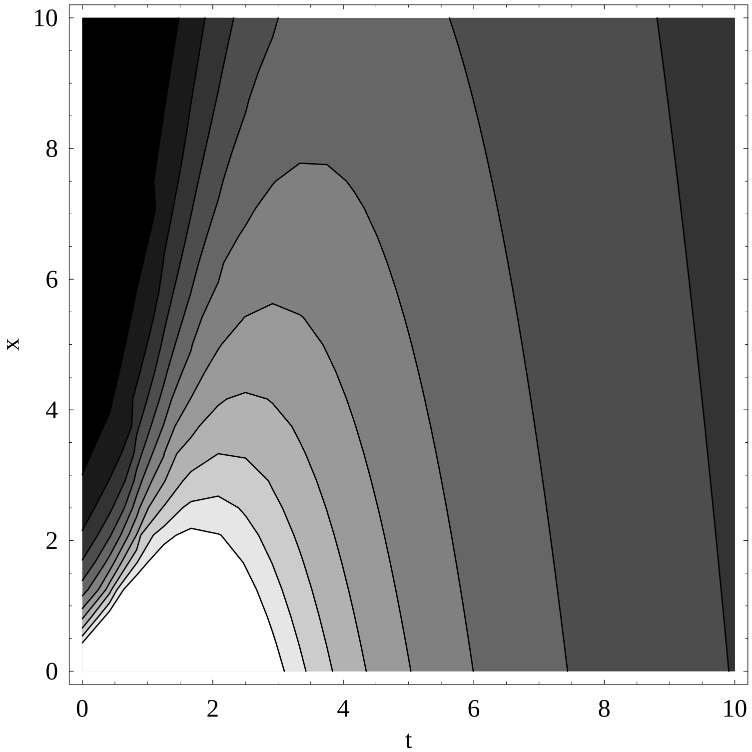

FIG. 2: The counter graph of $|\Psi(x,t)|^2$

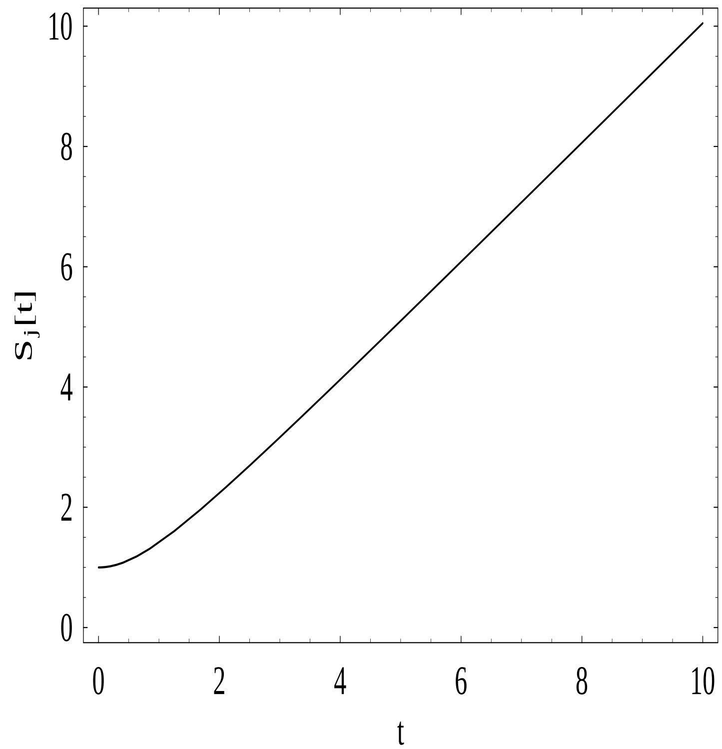

FIG. 3: The joint entropy of constant forces motion versus time